VECTORS

INTRODUCTION:-

In our real life situation we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively.

We have also come across physical quantities such as displacement, velocity, acceleration, momentum etc, which are of different type in comparison to above.

Consider the figure-1, where A, B, C are at a distance 4k.m. from P. If we start from P, then covering 4k.m. distance is not sufficient to describe the destination where we reach after the travel, So here the end point plays an important role giving rise the need of direction. So we need to study about direction of a quantity, along with magnitude.

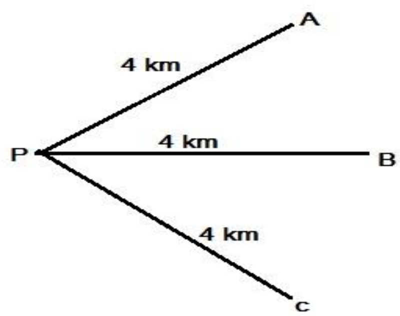

Fig - 1

OBJECTIVE

After completion of the topic you are able to :-

i) Define and distinguish between scalars and vectors.
ii) Represent a vector as directed line segment.
iii) Classify vectors in to different types.
iv) Resolve vector along two or three mutually perpendicular axes.
v) Define dot product of two vectors and explain its geometrical meaning.
vi) Define cross product of two vectors and apply it to find area of triangle and parallelogram.

<u>Expected background knowledge</u>

i) Knowledge of plane and co-ordinate geometry
ii) Trigonometry.

<u>Scalars and vectors</u>

All the physical quantities can be divided into two types.

i) Scalar quantity or Scalar.
ii) Vector quantity or Vector.

Scalar quantity: - The physical quantities which requires only magnitude for its complete specification is called as scalar quantities.

Examples: - Speed, mass, distance, velocity, volume etc.

Vector: - A directed line segment is called as vector.

Vector quantities:- A physical quantity which requires both magnitude & direction for its complete specification and satisfies the law of vector addition is called as vector quantities.
Examples: - Displacement, force, acceleration, velocity, momentum etc.

Representation of vector:- A vector is a directed line segment $\overrightarrow{AB}$where A is the initial point and B is the terminal point and direction is from A to B. (see fig-2).

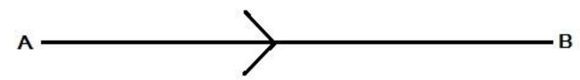

Fig - 2

Similarly $\overrightarrow{BA}$ is a directed line which represents a vector having initial point B and terminal point A.

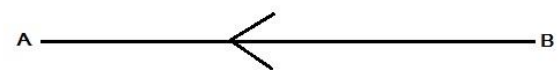

Fig - 3

Notation: - A vector quantity is always represented by an arrow (→) mark over it or by bar (−) over it. For example $\overrightarrow{AB}$. It is also represented by a single small letter with an arrow or bar mark over it. For example $\vec{a}$.

Magnitude of a vector: - Magnitude or modulus of a vector is the length of the vector. It is a scalar quantity.

Magnitude of $\overrightarrow{AB}$ = $|\overrightarrow{AB}|$= Length AB. = AB

<u>**Types of Vector: -**</u> Vectors are of following types.

1) **Null vector or zero vector or void vector: -** A vector having zero magnitude and arbitrary direction is called as a null vector and is denoted by $\vec{0}$.

Clearly, a null vector has no definite direction. If $\vec{a}$ = $\overrightarrow{AB}$, then $\vec{a}$ is a null (or zero) vector iff $|\vec{a}|$= 0 i.e. if $|\overrightarrow{AB}|$= 0

For a null vector initial and terminal points are same.

2) Proper vector: - Any non zero vector is called as a proper vector. If $|\vec{a}|\neq 0$ then $\vec{a}$ is a proper vector.

3) Unit vector : - A vector whose magnitude is unity is called a unit vector. Unit vectors are denoted by a small letter with ^ over it. For example $\hat{a}$. $|\hat{a}|$= 1

Note: - The unit vector along the direction of a vector $\vec{a}$ is given by

$$\boldsymbol{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$$

4) Co-initial vectors:- Vectors having the same initial point are called co-initial vector.

In figure-4, $\overrightarrow{OA}$,$\overrightarrow{OB}$,$\overrightarrow{OC}$,$\overrightarrow{OD}$ and $\overrightarrow{OE}$ are Co-initial vectors.

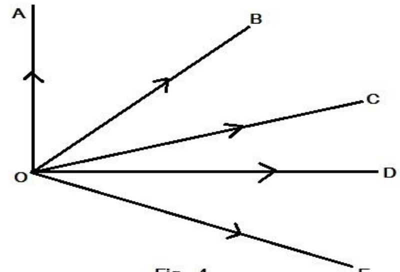

Fig - 4

5) Like and unlike vectors: - Vectors are said to be like if they have same direction and unlike if they have opposite direction.

6) Co-Linear vectors:- Vectors are said to be co-linear or parallal if they have the same line of action. In f figure-5 $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear.


Fig - 5

7) Parallel vectors: - Vectors are said to be parallel if they have same line of action or have line of action parallel to one another. In fig-6 the vectors are parallel to each other.

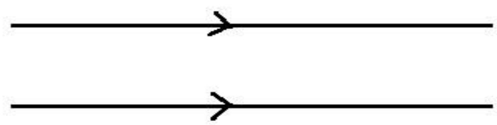
Fig - 6

8) Co-planner Vectors: - Vectors are said to be co-planner if they lies on the same plane. In fig-7 vector $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanner.

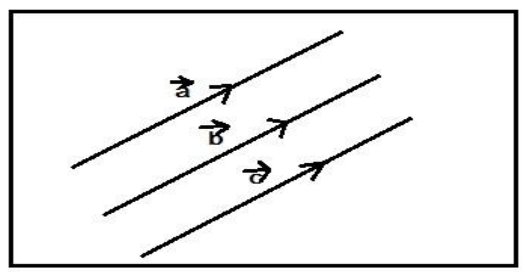
Fig - 7

9) Negative of a vector: - A vector having same magnitude but opposite in direction to that of a given vector is called negative of that vector. If $\overrightarrow{a}$ is any vector then negative vector of it is written as -$\overrightarrow{a}$ and $|\vec{a}|$ = $|-\vec{a}|$ but both have direction opposite to each other as shown in fig-8.

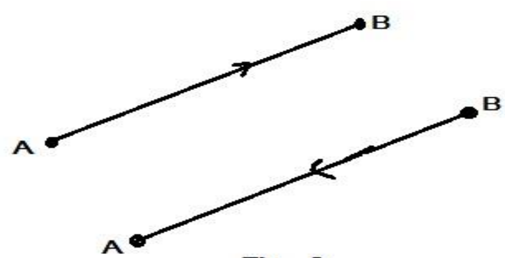

Fig - 8

10) Equal Vectors: - Two vectors are said to be equal if they have same magnitude as well as same direction.

Thus $\vec{a}$ = $\vec{b}$

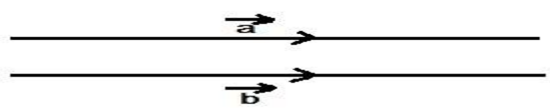
Fig - 9

Remarks:- Two vectors can not be equal

i) If they have different magnitude .

ii) If they have inclined supports.

iii) If they have different sense.

Vector operations

Addition of vectors: -

Triangle law of vector addition: - The law states that If two vectors are represented by the two sides of a triangle taken in same order their sum or resultant is represented by the 3rd side of the triangle with direction in reverse order.

As shown in figure-10 $\vec{a}$ and $\vec{b}$ are two vectors represented by two sides OA and AB of a triangle ABC in same order. Then the sum $\vec{a}+\vec{b}$ is represented by the third side OB taken in reverse order i.e. the vector $\vec{a}$ is represented by the directed segment $\overrightarrow{OA}$ and the vector $\vec{b}$ be the directed segment $\overrightarrow{AB}$, so that the terminal point A of $\vec{a}$ is the initial point of $\vec{b}$. Then $\overrightarrow{OB}$ represents the sum (or resultant) $(\vec{a}+\overrightarrow{b)}$. Thus $\overrightarrow{OB} = \vec{a}+\vec{b}$

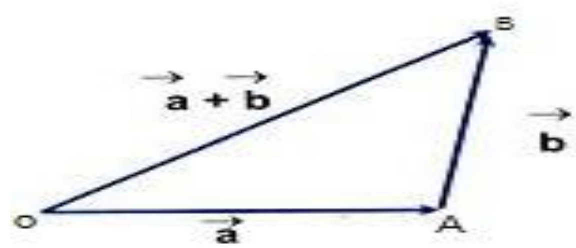

Fig – 10

Note-1 – The method of drawing a triangle in order to define the vector sum $(\vec{a}+\overrightarrow{b)}$ is called triangle law of addition of the vectors.

Note-2 – Since any side of a triangle is less than the sum of the other two sides

$|\overrightarrow{OB}| \neq |\overrightarrow{OA}| + |\overrightarrow{AB}|$

Parallelogram law of vector addition: - If $\vec{a}$ and $\vec{b}$ are two vectors represented by two adjacent side of a parallelogram in magnitude and direction, then their sum (resultant) is represented in magnitude and direction by the diagonal which is passing through the common initial point of the two vectors. As shown in fig-II if OA is $\vec{a}$ and AB is $\vec{b}$ then OB diagonal represent $\vec{a}+\vec{b}$.

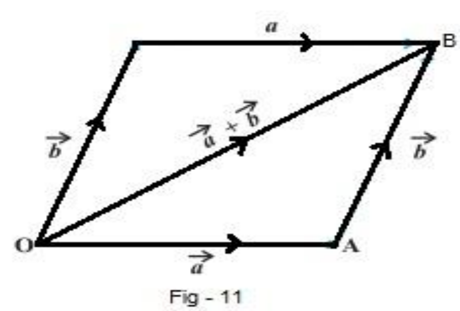

Fig - 11

i.e. $\vec{a}+\vec{b} = \overrightarrow{OA}+\overrightarrow{AB}$

Polygon law of vector addition: - If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are the four sides of a polygon in same order then their sum is represented by the last side of the polygon taken in opposite order as shown in figure-12.

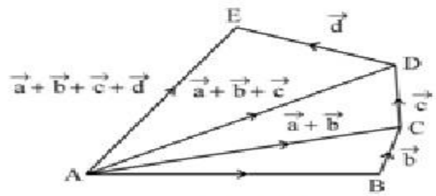

Fig - 12

Subtraction of two vectors

If $\vec{a}$ and $\vec{b}$ are two given vectors then the subtraction of $\vec{b}$ from $\vec{a}$ denoted by $\vec{a}$ - $\vec{b}$ is defined as addition of -$\vec{b}$ with $\vec{a}$. i.e. $\vec{a}$ - $\vec{b}$ =$\vec{a}$+ (-$\vec{b}$).

Properties of vector addition:- i) Vector addition is commutative i.e. if $\vec{a}$ & $\vec{b}$ are any two vectors then:-

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$

ii) Vector addition is associative i.e. if $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

iii) Existence of additive identity i.e. for any vector $\vec{a}$, $\vec{0}$ is the additive identity i.e. $\vec{0}+\vec{a}=\vec{a}+\vec{0}=\vec{a}$ where $\vec{0}$ is a null vector.

iv) Existence of additive Inverse :- If $\vec{a}$ is any non zero vector then -$\vec{a}$ is the additive inverse of $\vec{a}$, so that $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$

Multiplication of a vector by a scalar : -

If $\vec{a}$ is a vector and k is a nonzero scalar then the multiplication of the vector $\vec{a}$ by the scalar k is a vector denoted by k$\vec{a}$ or $\vec{a}$k whose magnitude $|k|$ times that of $\vec{a}$.

i.e k$\vec{a}$ = $|k|$ x $|\vec{a}|$

= k x $|\vec{a}|$ if $k \geq 0$.

= (-k) x $|\vec{a}|$ if k<0 .

The direction of k$\vec{a}$ is same as that of $\vec{a}$ if k is positive and opposite as that of $\vec{a}$ if k is negative.

k$\vec{a}$ and $\vec{a}$ are always parallel to each other.

Properties of scalar multiplication of vectors :-

If h and k are scalars and $\vec{a}$ and $\vec{b}$ are given vectors then

i) k $(\vec{a}+\vec{b})$ = k$\vec{a}$ + k$\vec{b}$

ii) (h+k)$\vec{a}$ = h$\vec{a}$ + k$\vec{a}$, (Distributive law)

iii) (hk)$\vec{a}$ = h(k$\vec{a}$), (Associative law)

iv) 1.$\vec{a}$= $\vec{a}$

v) 0. $\vec{a}$ = $\vec{0}$

Position Vector of a point

Let O be a fixed point called origin, let P be any other point, then the vector $\overrightarrow{OP}$ is called position vector of the point P relative to O and is denoted by $\vec{p}$.

As shown in figure-13, let AB be any vector , then applying triangle law of addition we have

$\overrightarrow{OA}$ + $\overrightarrow{AB}$ = $\overrightarrow{OB}$ where $\overrightarrow{\boldsymbol{OA}}$ **=** $\vec{a}$ and $\overrightarrow{OB}$ = $\vec{b}$

=> $\overrightarrow{\boldsymbol{AB}}$ **=** $\overrightarrow{\boldsymbol{OB}}$ **-** $\overrightarrow{\boldsymbol{OA}}$ **=** $\vec{b}$ - $\vec{a}$

= (Position vector of B)–(Position vector of A)

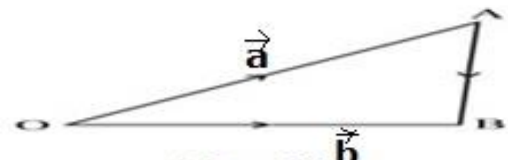

Fig - 13

Section Formula:- Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and P be a point on line segment AB , dividing it in the ration m:n. internally. Then the position vector of P i.e.$\vec{r}$ is given by the formula: $\vec{r} = \frac{m\vec{b}+n\,\vec{a}}{m+n}$

If P divides AB externally in the ratio m:n then $\vec{r} = \frac{m\vec{b}-n\,\vec{a}}{m-n}$

If P is the midpoint of AB then $\vec{r} = \frac{\vec{a}+\vec{b}}{2}$

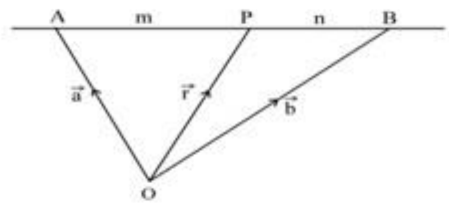

Fig - 14

Example-1 :- Prove that by vector method the medians of a triangle are concurrent.

Solution:- Let ABC be a triangle where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vector of A,B and C respectively. We have to show that the medians of this triangle are concurrent.

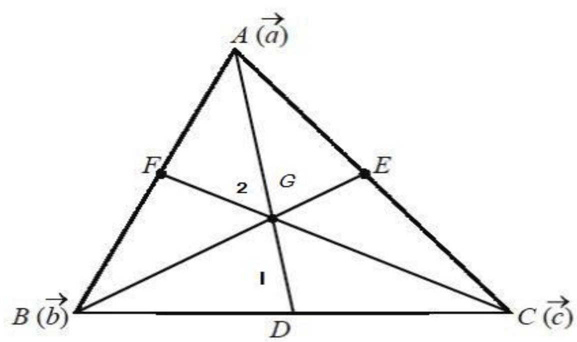

Fig - 15

Let AD , BE and CF are the three medians of the triangle.

Now as D be the midpoint of BC, so position vector of D i.e. $\vec{d} = \frac{\vec{b}+\vec{c}}{2}$.

Let G be any point of the median AD which divides AD in the ratio 2:1 . Then position vector of G is given

by $\vec{g} = \frac{2\vec{d}+\vec{a}}{2+1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1\vec{a}}{3}$ (by applying section formula)

$$\Rightarrow \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Let G' be a point which divides BE in the ratio 2:1 ,

Position vector of E is $\vec{e} = \frac{\vec{a}+\vec{c}}{2}$.

Then position vector of G' is given by $\overrightarrow{g'} = \frac{2\vec{e}+\vec{b}}{2+1} = \frac{2\left(\frac{\vec{a}+\vec{c}}{2}\right)+1\vec{b}}{3}$ (by applying section formula)

$$\Rightarrow \overrightarrow{g'} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

As position vector of a point is unique , so G = G' .

Similarly if we take G'' be a point on CF dividing it in 2:1 ratio then the position vector of G'' will be same as that of G.

Hence G is the one point where three median meet.

$\therefore$ The three medians of a triangle are concurrent. (proved)

Example2: - Prove that i) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (It is known as Triangle Inequality).

ii) $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$

iii) $|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$

Proof:- Let O,A and B be three points, which are not collinear and then draw a triangle OAB.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, then by triangle law of addition we have $\overrightarrow{OB} = \vec{a} + \vec{b}$

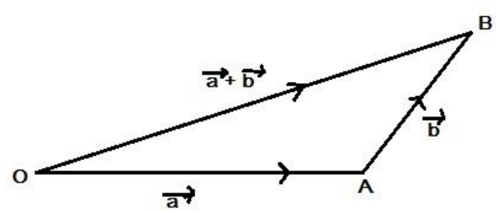

Fig - 16

From properties of triangle we know that the sum of any two sides of a triangle is greater than the third side.

- ⇨ OB < OA + AB
- ⇨ $|\overrightarrow{OB}| < |\overrightarrow{OA}| + |\overrightarrow{AB}|$
- ⇨ $|\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$ -------------------(1)

When O,A, B are collinear then

From figure-17 it is clear that

OB = OA + AB

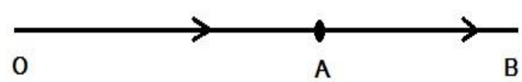

Fig-17

- ⇨ $|\overrightarrow{OB}| = |\overrightarrow{OA}| + |\overrightarrow{AB}|$
- ⇨ $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ -------------------(2)

From (1) and (2) we have,

$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (proved)

ii) $|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}|$ --------------(1)

But $|(\vec{a}-\vec{b})+\vec{b}| \le |\vec{a}-\vec{b}| + |\vec{b}|$ (From triangle inequality)--------(2)

From (1) and (2) we get $|\vec{a}| \le |\vec{a}-\vec{b}| + |\vec{b}|$

⇨ $|\vec{a}| - |\vec{b}| \le |\vec{a}-\vec{b}|$(proved)

iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \le |\vec{a}| + |-\vec{b}|$ (From triangle inequality)

$= |\vec{a}| + |\vec{b}|$ (as $|-\vec{b}| = |\vec{b}|$)

$|\vec{a}-\vec{b}| \le |\vec{a}| + |\vec{b}|$ (proved)

<u>Components of vector in 2D</u>

Let XOY be the co-ordinate plane and P(x,y) be any point in this plane.

The unit vector along direction of X axis i.e. $\overrightarrow{OX}$ is denoted by $\hat{\imath}$.

The unit vector along direction of Y axis i.e. $\overrightarrow{OY}$ is denoted by $\hat{\jmath}$.

Then from figure-18 it is clear that $\overrightarrow{OM} = x\hat{\imath}$ and $\overrightarrow{ON} = y\hat{\jmath}$.

So, the position vector of P is given by

$$\boxed{\overrightarrow{OP} = \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}}$$

And OP $=|\overrightarrow{OP}| = r = \sqrt{x^2+y^2}$

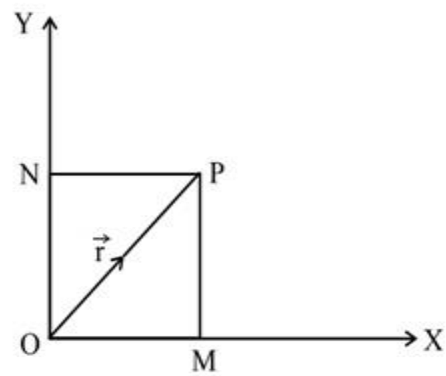

Fig-18

<u>Representation of vector in component form in 2D</u>

If $\overrightarrow{AB}$ is any vector having end points A(x_1, y_1) and B (x_2, y_2) , then it can be represented by

$\overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$

Components of vector in 3D

Let P(x,y,z) be a point in space and $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ be the unit vectors along X axis , Y axis and Z axis respectively. (as shown in fig-19)

Then the position vector of P is given by

$\overrightarrow{OP}$ = $x\,\hat{\imath}$ + y $\hat{\jmath}$ + $\hat{k}$, The vectors x$\hat{\imath}$, y $\hat{\jmath}$, z$\hat{k}$ are called the components of $\overrightarrow{OP}$ along x–axis, y-axis and z-axis respectively.

And OP =|$\overrightarrow{OP}$| = $\sqrt{x^2+y^2+z^2}$

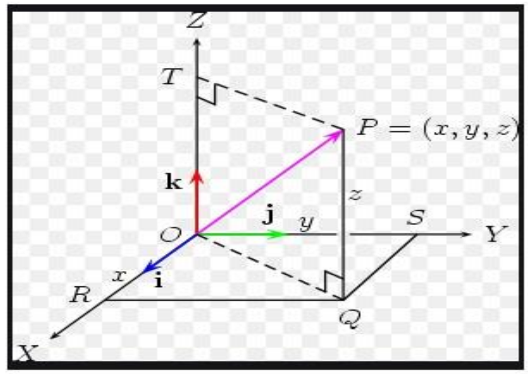

Fig-19

Addition and scalar multiplication in terms of component form of vectors: -

For any vector $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

i) $\vec{a} + \vec{b} = (a_1 + b_1)\,\hat{\imath} + (a_2 + b_2)\hat{\jmath} + (a_3 + b_3)\hat{k}$

ii) $\vec{a} - \vec{b} = (a_1 - b_1)\,\hat{\imath} + (a_2 - b_2)\hat{\jmath} + (a_3 - b_3)\hat{k}$

iii) k $\vec{a} = ka_1\hat{\imath} + ka_2\hat{\jmath} + ka_3\hat{k}$, where K is a scalar.

iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$\Leftrightarrow a_1=b_1,\ a_2=b_2,\ a_3=b_3$

Representation of vector in component form in 3-D & Distance between two points:

If $\overrightarrow{AB}$ is any vector having end points A(x_1, y_1, z_1) and B (x_2, y_2, z_2) , then it can be represented by

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$=(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$$

$$= \boldsymbol{(x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}}$$

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 3:-

Show that the points A(2,6,3) , B(1,2,7) and c(3,10,-1) are collinear.

Solution:- From given data Position vector of A, $\overrightarrow{OA} = 2\,\hat{\imath} + 6\hat{\jmath} + 3\,\hat{k}$.

Position vector of B, $\overrightarrow{OB} = \hat{\imath} + 2\hat{\jmath} + 7\hat{k}$

Position vector of C, $\overrightarrow{OC} = 3\,\hat{\imath} + 10\hat{\jmath} - \hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1-2)\,\hat{\imath} + (2-6)\,\hat{\jmath} + (7-3)\,\hat{k} = -\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3-2)\,\hat{\imath} + (10 - 6)\,\hat{\jmath} + (-1-3)\,\hat{k} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$.

$= -(-\hat{\imath} - 4\hat{\jmath} + 4\hat{k}) = -\overrightarrow{AB}$

⇨ $\overrightarrow{AB} \parallel \overrightarrow{AC}$ or collinear.

∴ They have same support and common point A.
As 'A' is common to both vector , that proves A,B and C are collinear.

Example-4: - Prove that the points having position vector given by $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ form a right angled triangle. [2009(w)]

S0lution :- Let A,B and C be the vertices of a triangle with position vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ respectively

Then . $\overrightarrow{AB}$ = Position vector of B – Position vector of A.

$= (1-2)\,\hat{\imath} + (-3 - (-1))\,\hat{\jmath} + (-5-1)\,\hat{k} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$.

$\overrightarrow{BC}$ = Position vector of C – Position vector of B.

$= (3-1)\,\hat{\imath} + (-4 - (-3))\,\hat{\jmath} + (-4-(-5))\,\hat{k} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$.

$\overrightarrow{AC}$ = Position vector of C – Position vector of A.

$= (3-2)\,\hat{\imath} + (-4 - (-1))\,\hat{\jmath} + (-4-1)\,\hat{k} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$.

Now AB = $|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

BC = $|\overrightarrow{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

AC = $|\overrightarrow{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

From above $BC^2 + AC^2 = 6+35 = 41 = AB^2$.

Hence ABC is a right angled triangle.

Example-5 :- Find the unit vector in the direction of the vector $\vec{a} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k}$. (2017-W)

Ans:- The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{\imath} - 4\hat{\jmath} + \hat{k}}{\sqrt{3^2+(-4)^2+1^2}} = \frac{3\hat{\imath} - 4\hat{\jmath} + \hat{k}}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}\hat{\imath} - \frac{4}{\sqrt{26}}\hat{\jmath} + \frac{1}{\sqrt{26}}\hat{k}.$$

Example-6 :- Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = = \hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans:- Let $\vec{r} = \vec{a} + \vec{b} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + (\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 2\hat{\imath} + 2\hat{k}$.

Unit vector along direction of $\vec{a} + \vec{b}$ is given by $= \frac{\vec{r}}{|\vec{r}|} = \frac{2\hat{\imath} + 2\hat{k}}{\sqrt{2^2+2^2}} = \frac{2\hat{\imath} + 2\hat{k}}{\sqrt{8}} = \frac{2}{\sqrt{8}}\hat{\imath} + \frac{2}{\sqrt{8}}\hat{k}$

$= \frac{2}{2\sqrt{2}}\hat{k}\hat{\imath} + \frac{2}{2\sqrt{2}}\hat{\jmath} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}.$

Angle between the vectors:-

As shown in figure-20 angle between two vectors $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ can be determined as follows.

Let $\overrightarrow{OB}$be a vector parallel to $\overrightarrow{RS}$ and $\overrightarrow{OA}$ is a vector parallel to $\overrightarrow{PQ}$ such that $\overrightarrow{OB}$and $\overrightarrow{OA}$ intersect each other.

Then θ = ∟AOB = angle between $\overrightarrow{RS}$ and $\overrightarrow{PQ}$.

If θ = 0 then vectors are said to be parallel.

If θ = $\frac{\pi}{2}$ then vectors are said to be orthogonal or perpendicular.

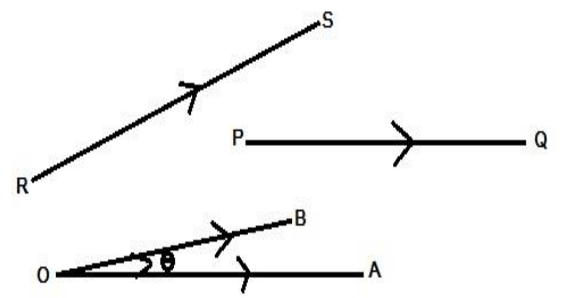

Fig-20

Dot Product or Scalar product of vectors

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are, a and b respectively denoted by $\vec{a}.\vec{b}$ is defined as the scalar abcosθ , where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$.

$$\boxed{\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = a\, b \cos\theta}$$

Geometrical meaning of dot product

In figure21(a), $\vec{a}$ and $\vec{b}$ are two vectors having θ angle between them. Let M be the foot of the perpendicular drawn from B to OA.

Then OM is the Projection of $\vec{b}$ on $\vec{a}$ and from figure-21(a) it is clear that ,

|OM| = |OB| cosθ = |$\vec{b}$| cosθ.

Now $\vec{a}.\vec{b}$ = |$\vec{a}$| (|$\vec{b}$| cosθ) = |$\vec{a}$| x projection of $\vec{b}$ on $\vec{a}$

which gives projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Similarly we can write $\vec{a}.\vec{b}$ = |$\vec{a}$| |$\vec{b}$| cosθ

=|$\vec{b}$| (|$\vec{a}$| cosθ) = |$\vec{b}$| projection of $\vec{a}$ on $\vec{b}$.

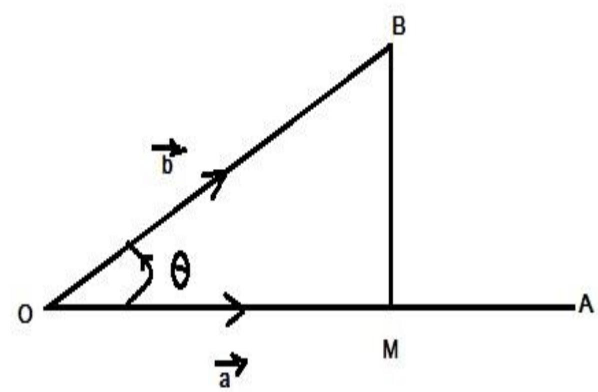

Fig-21

(a)

Similarly, let us draw a perpendicular from A on OB and let N be the foot of the perpendicular in fig-21(b).

Then ON = Projection of $\vec{a}$ on $\vec{b}$

and ON = OA$\cos\theta$ = I$\vec{a}$I $\cos\theta$.

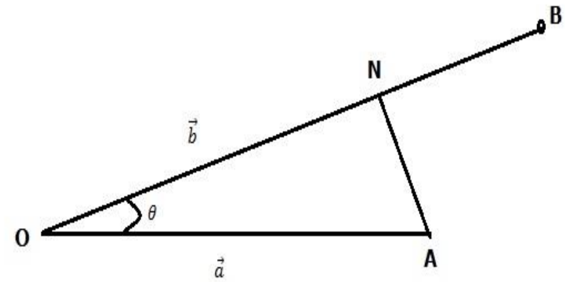

Fig-21(b)

Properties of Dot product

i) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)

ii) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (Distributive)

iii) If $\vec{a}$ II $\vec{b}$, then $\vec{a}.\vec{b}$ = ab { as θ = 0 in this case cos 0 = 1 }

In particular $(\vec{a})^2 = \vec{a}\,.\,\vec{a}$ = I $\vec{a}$I^2

$$\hat{\imath}.\,\hat{\imath} = \hat{\jmath}.\,\hat{\jmath} = \hat{k}.\,\hat{k} = 1$$

iv) If $\vec{a} \perp \vec{b}$, then $\vec{a}.\vec{b}$ = 0. { as θ = 90º in this case cos90º= 0 }

In particular $\hat{\imath}.\,\hat{\jmath} = \hat{\jmath}.\,\hat{k} = \hat{k}.\,\hat{\imath} = 0 = \hat{\jmath}.\hat{\imath} = \hat{k}.\hat{\jmath} = \hat{\imath}.\hat{k}$

v) $\vec{a}.\vec{0} = \vec{0}.\vec{a} = 0$

vi) $(\vec{a}+\vec{b})\,.\,(\vec{a}-\vec{b})$ = I$\vec{a}$I^2- I$\vec{b}$I^2= $a^2 - b^2$ {Where I$\vec{a}$ I = a and I$\vec{b}$ I = b }

viii) Work done by a Force:- The work done by a force $\vec{F}$ acting on a body causing displacement $\vec{d}$ is given by W = $\vec{F}.\vec{d}$

Dot product in terms of rectangular components

For any vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ we have,

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (by applying distributive(ii) , (iii) and (iv) successively)

Angle between two non zero vectors

For any two non zero vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, having θ is the angle between them we have,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \hat{a}.\hat{b} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$ **(In terms of components.)**

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Condition of Perpendicularity: -

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are perpendicular to each other $\Leftrightarrow \boldsymbol{a_1b_1 + a_2b_2 + a_3b_3 = 0}$

Condition of Parallelism :-

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are parallel to each other $\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$)

<u>Scalar & vector projections of two vectors (Important formulae)</u>

Scalar Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Vector Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\hat{a} = [\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}]\vec{a}$

Scalar Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b}$

Examples: -

Q.- 7. Find the value of p for which the vectors $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$, $\hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are perpendicular to each other.

Solution:- Let $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$.

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 9$

$b_1 = 1$, $b_2 = p$ & $b_3 = 3$

Given $\vec{a} \perp \vec{b} \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

$\Rightarrow 3.1+2.p+9.3 = 0$

$\Rightarrow 3 + 2p + 27 = 0$

$\Rightarrow 2p = -30$

$\Rightarrow p = -15$ (Ans)

Q-8 Find the value of p for which the vectors $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$, $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are parallel to each other. (2014-W)

Solution:- Given $\vec{a} \parallel \vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \Leftrightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$ { Taking 1st two terms}

$\Leftrightarrow 3 = \frac{2}{p} \Leftrightarrow p = \frac{2}{3}$ (Ans) {Note:- any two expression may be taken for finding p.}

Q-9 Find the scalar product of $3\hat{\imath} - 4\hat{\jmath}$ and $-2\hat{\imath} + \hat{\jmath}$. (2015-S)

Solution:- $(3\hat{\imath} - 4\hat{\jmath}) . (-2\hat{\imath} + \hat{\jmath}) = (3\times(-2))+((-4)\times 1) = (-6)+(-4) = -10$

Q-10 Find the angle between the vectors $5\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $6\hat{\imath} - 8\hat{\jmath} - \hat{k}$. (2015-W)

Solution:- Let $\vec{a} = 5\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = 6\hat{\imath} - 8\hat{\jmath} - \hat{k}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then $\theta = cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}}\right)$

$= cos^{-1}\left(\frac{5.6+3.(-8)+4.(-1)}{\sqrt{5^2+3^2+4^2}\sqrt{6^2+(-8)^2+(-1)^2}}\right) = cos^{-1}\left(\frac{30-24-4}{\sqrt{50}\sqrt{101}}\right) = cos^{-1}\left(\frac{2}{\sqrt{50}\sqrt{101}}\right)$

Q-11 Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where,

$\vec{a} = \hat{\imath} - \hat{\jmath} - \hat{k}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 3\hat{k}$. { 2013-W ,2017-W, 2017-S}

Solution:- Scalar Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{1.3+(-1).1+(-1).3}{\sqrt{3^2+1^2+3^2}} = \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$

Vector Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{1.3+(-1).1+(-1).3}{(\sqrt{3^2+1^2+3^2})^2}$ $(3\hat{\imath} + \hat{\jmath} + 3\hat{k})$

$= \frac{3-1-3}{19}(3\hat{\imath} + \hat{\jmath} + 3\hat{k}) = \frac{-1}{19}(3\hat{\imath} + \hat{\jmath} + 3\hat{k})$

Q-12 Find the scalar and vector projection of $\vec{b}$ on $\vec{a}$ where,

$\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$. { 2015-S}

Solution: - Scalar Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{a}.\vec{b}}{|\vec{a}|} = \frac{3.2+1.3+(-2).(-4)}{\sqrt{3^2+1^2+(-2)^2}} = \frac{6+3+8}{\sqrt{14}} = \frac{17}{\sqrt{14}}$

Vector Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{3.2+1.3+(-2).(-4)}{(\sqrt{3^2+1^2+(-2)^2})^2}(3\hat{\imath} + \hat{\jmath} - 2\hat{k})$

$= \frac{17}{14}(3\hat{\imath} + \hat{\jmath} - 2\hat{k})$.

Q-13 If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, then prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Proof:- Given $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

⇨ $(\vec{a}.\vec{b}) - (\vec{a}.\vec{c}) = \vec{0}$ => $\vec{a}.(\vec{b} - \vec{c}) = \vec{0}$ { applying distributive property}

Dot product of above two vector is zero indicates the following conditions

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{o}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

⇨ $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ (proved)

Example:-14 Find the work done by the force $\vec{F} = \hat{\imath} + \hat{\jmath} - \hat{k}$. acting on a particle if the particle is displace A from A(3,3,3) to B(4,4,4).

Ans:- Let O be the origin, then

Position vector of A $\overrightarrow{OA} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$

Position vector of B $\overrightarrow{OB} = 4\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$

Then displacement is given by, $\vec{d} = \overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) = (4\hat{\imath}+4\hat{\jmath}+ 4\hat{k}) - (3\hat{\imath}+3\hat{\jmath}+3\hat{k}) = \hat{\imath}+\hat{\jmath}+ \hat{k}$.

So work done by the force $W = \vec{F}.\vec{d} = \vec{F}.\overrightarrow{AB} = (\hat{\imath}+\hat{\jmath}- \hat{k}).(\hat{\imath}+\hat{\jmath}+ \hat{k})$

$= 1.1+1.1+(-1).1 = 1$ units

Example:-15 If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them then prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}- \hat{b}|$$

Proof: - $(|\hat{a}- \hat{b}|)^2 = (\hat{a}- \hat{b}).(\hat{a}- \hat{b}) = (\hat{a}.\hat{a}) - (\hat{a}.\hat{b}) -(\hat{b}.\hat{a}) + (\hat{b}.\hat{b})$ { Distributive property}

$= (|\hat{a}|)^2 - (\hat{a}.\hat{b}) - (\hat{a}.\hat{b}) + (|\hat{b}|)^2$ {commutative property}

$= 1^2 - 2\,\hat{a}.\hat{b} + 1^2$ { as $\hat{a}$ and $\hat{b}$ are unit vectors so their magnitudes are 1}

$= 2 - 2\,\hat{a}.\hat{b} = 2(1 - \hat{a}.\hat{b})$

$= 2(1 - |\hat{a}|.|\hat{b}|\cos\theta)$ { as θ is the angle between $\hat{a}$ and $\hat{b}$ }

$= 2(1 – 1.1.\cos\theta)$

$= 2(1 – \cos\theta) = 2.2\,sin^2\frac{\theta}{2}$

Taking square root of both sides we have $|\hat{a}- \hat{b}| = 2\sin\frac{\theta}{2}$

⇨ $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}- \hat{b}|$ (proved)

Example:-16 If the sum of two unit vectors is a unit vector. Then show that the magnitude of their difference is $\sqrt{3}$.

Proof:-$\hat{a}$,$\hat{b}$ and $\hat{c}$ are three unit vectors such that $\hat{a}+ \hat{b} = \hat{c}$

Squaring both sides we have,

=> $(|\hat{a}+ \hat{b}|)^2 = (|\hat{c}|)^2$

=> $(|\hat{a}|)^2 + (|\hat{b}|)^2 + 2\,\hat{a}.\hat{b} = 1^2$

=> $1^2 + 1^2 + 2|\hat{a}||\hat{b}|\cos\theta = 1$ { where θ is the angle between $\hat{a}$ and $\hat{b}$ }

=> $1 + 1 + 2\cos\theta = 1$

=> $2\cos\theta = -1$

=> $\cos\theta = \frac{-1}{2}$

Now we have to find the magnitude of their difference i.e $\hat{a}- \hat{b}$.

So $(|\hat{a}- \hat{b}|)^2 = (|\hat{a}|)^2 + (|\hat{b}|)^2 - 2\,\hat{a}.\hat{b} = 1^2 + 1^2 - 2|\hat{a}||\hat{b}|\cos\theta$

$= 2- 2\cos\theta = 2 - 2(\frac{-1}{2}) = 2 – (-1) = 3$

$\therefore |\hat{a}- \hat{b}| = \sqrt{3}$ (Proved)